Completely Regular Fuzzifying Topological Spaces

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Abstract

Some of the properties of the completely regular fuzzifying topological spaces are investigated. It is shown that a fuzzifying topology τ is completely regular iff it is induced by some fuzzy uniformity or equivalently by some fuzzifying proximity. Also, τ is completely regular iff it is generated by a family of probabilistic pseudometrics.

Key words and phrases: Fuzzifying topology, Fuzzifying proximity, fuzzy uniformity, probabilistic pseudometric.

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Introduction

The fuzzifying topologies were introduced by M. Ying in [15]. A classical topology is a special case of a fuzzifying topology. In a fuzzifying topology τ on a set X, every subset A of X has a degree $\tau(A)$ of belonging to τ, $0 \leq \tau(A) \leq 1$. In [1] we defined the degrees of compactness, of local compactness, Hausdorffnes e.t.c. in a fuzzifying topological space (X, τ). We also gave the notion of convergence of nets and filters and introduced the fuzzifying proximities. Every fuzzifying proximity δ induces a fuzzifying topology τ_δ. In [4] we studied the level classical topologies $\tau^\theta, 0 \leq \theta < 1$, corresponding to a fuzzifying topology τ. In the same paper we studied connectedness and local connectedness in fuzzifying topological spaces as well as the so called sequential fuzzifying topologies. In [3] we introduced the fuzzifying syntopogenous structures. We also proved that every fuzzy uniformity $\mathcal{U}$, as it is defined by Lowen in [8], induces a fuzzifying proximity $\delta_\mathcal{U}$ and that , for every fuzzifying proximity δ, there exists at least one fuzzy uniformity $\mathcal{U}$ with $\delta = \delta_\mathcal{U}$.
In this paper, we continue with the investigation of fuzzifying topologies. In particular we study the completely regular fuzzifying topologies, i.e those fuzzifying topologies τ for which each level topology τ^θ is completely regular. As in the classical case, we prove that, for a fuzzifying topology τ on X, the following properties are equivalent: (1) τ is completely regular; (2) τ is uniformizable, i.e. it is induced

by some fuzzy uniformity; (3) τ is proximizable, i.e. it is induced by some fuzzifying proximity; (4) τ is generated by a family of so called probabilistic pseudometrics on X. We also give a characterization of completely regular fuzzifying spaces in terms of continuous functions. Many Theorems on classical topologies follow as special cases of results obtained in the paper.

1 Preliminaries

A fuzzifying topology on a set X (see [15]) is a map $\tau : 2^X \to [0,1]$, (where 2^X is the power set of X) satisfying the following conditions:
(FT1) $\tau(X) = \tau(\emptyset) = 1$.
(FT2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$.
(FT3) $\tau(\bigcup A_i) \geq \inf_i \tau(A_i)$.

If τ is a fuzzifying topology on X and $x \in X$, then the τ-neighborhood system of x is the function

$$N_x = N_x^\tau : 2^X \to [0,1], N_x(A) = \sup\{\tau(B) : x \in B \subset A\}.$$

By ([15], Lemma 3.2) we have that $\tau(A) = \inf_{x \in A} N_x(A)$.

Theorem 1.1 *([15], I, Theorem 3.2). If τ is a fuzzifying topology on a set X, then the map $x \to N_x = N_x^\tau$, from X to the fuzzy power set $\mathcal{F}(2^X)$ of 2^X, has the following properties:*
(FN1) $N_x(X) = 1$ and $N_x(A) = 0$ if $x \notin A$.
(FN2) $N_x(A_1 \bigcap A_2) = N_x(A_1) \wedge N_x(A_2)$.
(FN3) $N_x(A) \leq \sup_{x \in D \subset A} \inf_{y \in D} N_y(D)$.
Conversely, if a map $x \to N_x$, from X to $\mathcal{F}(2^X)$, satisfies $(FN1)-(FN3)$, then the map

$$\tau : 2^X \to [0,1], \tau(A) = \inf_{x \in A} N_x(A),$$

is a fuzzifying topology and $N_x = N_x^\tau$ for every $x \in X$.

Let now (X, τ) be a fuzzifying topological space. To every subset A of X corrersponds a fuzzy subset $\bar{A} = \bar{A}^\tau$ of X defined by $\bar{A}(x) = 1 - N_x(A^c)$. A function f, from a fuzzifying topological space (X, τ_1) to another one (Y, τ_2), is said to be continuous at some $x \in X$ (see [2]) if $N_x(f^{-1}(A)) \geq N_{f(x)}(A)$ for every subset A of Y. If f is continuous at every point of X, then it is said $(\tau_1, \tau_2)-$continuous. As it is shown in [2], f is continuous iff $\tau_2(A) \leq \tau_1(f^{-1}(A))$ for every subset A of Y. For $f : X \to Y$ a function and τ a fuzzifying topology on $Y, f^{-1}(\tau)$ is defined to be the weakest fuzzifying topology on X for which f is continuous. By [2], $f^{-1}(\tau)$ is given by the neighborhood structure $N_x(A) = N_{f(x)}(Y \setminus f(A^c))$. If $(\tau_i)_{i \in I}$ is a family of fuzzifying topologies on X, we will denote by $\bigvee_{i \in I} \tau_i$, or by $\sup \tau_i$, the weakest of all fuzzifying topologies on X which are finer than each τ_i. As it is proved in [2], $\bigvee_{i \in I} \tau_i$ is given by the neighborhood structure

$$N_x(A) = \sup\{\inf_{i \in J} N_x^{\tau_i}(A_i) : x \in \bigcap_{i \in J} A_i \subset A\},$$

where the infimum is taken over the family of all finite subsets J of I and all $A_i \subset X, i \in J$. For Y a subset of a fuzzifying topological space $(X, \tau), \tau|_Y$ will be the fuzzifying topology induced on Y by τ, i.e. the fuzzifying topology $f^{-1}(\tau)$ where $f : Y \to X$ is the inclusion map. For a family $(X_i, \tau_i)_{i \in I}$ of fuzzifying topological spaces, the product fuzzifying topology $\tau = \prod \tau_i$ on $X = \prod X_i$ is the weakest fuzzifying topology on X for which each projection $\pi_i : X \to X_i$ is continuous. Thus $\tau = \bigvee_i \pi_i^{-1}(\tau_i)$ and it is given by the neighborhood structure

$$N_x(A) = \sup\{\inf_{i \in J} N_{x_i}(A_i) : x \in \bigcap_{i \in J} \pi_i^{-1}(A_i) \subset A\},$$

where the supremum is taken over the family of all finite subsets J of I and $A_i \subset X_i$, for $i \in J$ (see [2]).

The degree of convergence to an $x \in X$, of a net (x_δ) in a fuzzifying topological space (X, τ), is the number $c(x_\delta \to x) = c^\tau(x_\delta \to x)$ defined by

$$c(x_\delta \to x) = \inf\{1 - N_x(A) : A \subset X, (x_\delta) \quad \text{frequently in } A^c\}.$$

As it is shown in [4], for $A \subset X$ and $x \in X$, we have

$$\bar{A}(x) = \max\{c(x_\delta \to x) : (x_\delta) \text{ net in } A\}.$$

The degree of Haudsdorffness of X (see [2]) is defined by

$$T_2(X) = 1 - \sup_{x \neq y} \sup\{c(x_\delta \to x) \wedge c(x_\delta \to y : (x_\delta) \text{ net in } X\}.$$

Also, the degree of X being T_1 is defined by

$$T_1(X) = \inf_x \inf_{y \neq x} \sup\{N_x(B) : y \notin B\}.$$

Let now (X, τ) be a fuzzifying topological space. For each $0 \leq \theta < 1$, the family $B_\theta^\tau = \{A \subset X : \tau(A) > \theta\}$ is a base for a classical topology τ^θ on X (see [3]). It is easy to see that a subset B of X is a τ^θ-neighborhood of x iff $N_x(B) > \theta$. By [4], $T_2(X)$ (resp. $T_1(X)$) is the supremum of all $0 \leq \theta < 1$ for which τ^θ is T_2 (resp. T_1). Also, for $\tau = \vee \tau_i$, we have that $\tau^\theta = \sup_i \tau_i^\theta$ (see [3], Theorem 3.5). If $\tau = \prod \tau_i$ is a product fuzzifying topology, then $\tau^\theta = \prod \tau_i^\theta$ (see [3], Theorem 3.5). If Y is a subspace of (X, τ) and $\tau_1 = \tau|Y$, then $\tau_1^\theta = \tau^\theta|Y$. By [3], Theorem 3.10, for a fuzzifying topological space (X, τ), $co(X)$ coincides with the supremum of all $0 < \theta < 1$ for which $\tau^{1-\theta}$ is compact.

Next we will recall the notion of a fuzzifying proximity given in [2]. A fuzzifying proximity on a set X is a map $\delta : 2^X \times 2^X \to [0, 1]$ satisfying the following conditions:

(FP1) $\delta(A, B) = 1$ if the A, B are not disjoint.

(FP2) $\delta(A, B) = \delta(B, A)$.

(FP3) $\delta(\emptyset, B) = 0$.

(FP4) $\delta(A_1 \cup A_2, B) = \delta(A_1, B) \vee \delta(A_2, B)$.

(FP5) $\delta(A, B) = \inf\{\delta(A, D) \vee \delta(D^c, B) : D \subset X\}$.

Every fuzzifying proximity δ induces a fuzzifying topology τ_δ given by the neighborhood structure $N_x(A) = 1 - \delta(x, A^c)$. A fuzzifying proximity δ_1 is said to be finer than another one δ_2 if $\delta_1(A,B) \leq \delta_2(A,B)$ for all subsets A, B of X. For $f : X \to Y$ a function and δ a fuzzifying proximity on Y, the function

$$f^{-1}(\delta) : 2^X \times 2^X \to [0,1], f^{-1}(\delta)(A,B) = \delta(f(A), f(B)),$$

is a fuzzifying proximity on X (see [2]) and it is the weakest of all fuzzifying proximities δ_1 on X for which f is (δ_1, δ)-proximally continuous, i.e. it satisfies $\delta_1(A,B) \leq \delta(f(A), f(B))$ for all subsets A, B of X. As it is shown in [2], $\tau_{f^{-1}(\delta)} = f^{-1}(\tau_\delta)$.
Let now $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X. We will denote by $\delta = \bigvee_\lambda \delta_\lambda$, or by $\sup \delta_\lambda$, the weakest fuzzifying proximity on X which is finer than each δ_λ. By [2], Theorem 8.10, δ is given by

$$\delta(A,B) = \inf\{\sup_{i,j} \inf_{\lambda\in\Lambda} \delta_\lambda(A_i, B_j)\},$$

where the infimum is taken over all finite collections $(A_i), (B_j)$ of subsets of X with $A = \bigcup A_i, B = \bigcup B_j$. Moreover, $\tau_\delta = \bigvee \tau_{\delta_\lambda}$ (see [2]).
Finally we will recall the definition of a fuzzy uniformity introduced by Lowen in [8]. For a set X, let Ω_X be the collection of all functions $\alpha : X \times X \to [0,1]$ such that $\alpha(x,x) = 1$ for all $x \in X$. For α, β in Ω_X the $\alpha \wedge \beta, \alpha \circ \beta$ and α^{-1} are defined by $\alpha\wedge\beta(x,y) = \alpha(x,y)\wedge\beta(x,y), \alpha\circ\beta(x,y) = \sup_z \beta(x,z)\wedge\alpha(z,y), \alpha^{-1}(x,y) = \alpha(y,x)$. If $\alpha = \alpha^{-1}$, then α is called symmetric. A fuzzy uniformity on X is a non-empty subset $\mathcal{U}$ of Ω_X satisfying the following conditions :
(FU1) If $\alpha, \beta \in \mathcal{U}$, then $\alpha \wedge \beta \in \mathcal{U}$.
(FU2) If $\alpha \in \mathcal{U}$ is such that, for every $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \leq \alpha + \epsilon$, then $\alpha \in \mathcal{U}$.
(FU3) For each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \circ \beta \leq \alpha + \epsilon$.
(FU4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$.
A subset $\mathcal{B}$, of a fuzzy uniformity $\mathcal{U}$, is a base for $\mathcal{U}$ if, for each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$. It is easy to see that, for a subset $\mathcal{B}$ of Ω_X, the following are equivalent :
(1) $\mathcal{B}$ is a base for a fuzzy uniformity on X.
(2) (a) If $\alpha, \beta \in \mathcal{B}$ and $\epsilon > 0$, then there exists $\gamma \in \mathcal{B}$ with $\gamma \leq \alpha \wedge \beta + \epsilon$.

(b) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \circ \beta \leq \alpha + \epsilon$.

(c) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \leq \alpha^{-1} + \epsilon$.

In case (2) is satisfied, the fuzzy uniformity $\mathcal{U}$ for which $\mathcal{B}$ is a base consists of all $\alpha \in \Omega_X$ such that, for each $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ with $\beta \leq \alpha + \epsilon$.
By [3], every fuzzy uniformity $\mathcal{U}$ on X induces a fuzzifying proximity $\delta_\mathcal{U}$ defined by

$$\delta_\mathcal{U}(A,B) = \inf_{\alpha\in\mathcal{U}} \sup_{x\in A, y\in B} \alpha(x,y).$$

In case $\mathcal{B}$ is a base for $\mathcal{U}$, then

$$\delta_\mathcal{U}(A,B) = \inf_{\alpha\in\mathcal{B}} \sup_{x\in A, y\in B} \alpha(x,y).$$

Every fuzzy uniformity $\mathcal{U}$ induces a fuzzifying topology $\tau_{\mathcal{U}}$ given by the neighborhood structure

$$N_x(A) = 1 - \delta_{\mathcal{U}}(x, A^c) = 1 - \inf_{\alpha \in \mathcal{U}} \sup_{y \notin A} \alpha(x, y).$$

For every fuzzifying proximity δ there exists at least one compatible fuzzy uniformity, i.e. a fuzzy uniformity $\mathcal{U}$ with $\delta_{\mathcal{U}} = \delta$ (see [3], Theorem 11.4).

2 Probabilistic Pseudometrics

A fuzzy real number is a fuzzy subset u of the real numbers $\mathbf{R}$ which is increasing, left continuous, and such that $\lim_{t \to +\infty} u(t) = 1, \lim_{t \to -\infty} u(t) = 0$. A fuzzy real number u is said to be non-negative if $u(t) = 0$ if $t \leq 0$. We will denote by $\mathbf{R}_\phi^+$ the collection of all non-negative fuzzy real numbers. To every real number r corresponds a fuzzy real number $\bar{r}$, where $\bar{r}(t) = 0$ if $t \leq r$ and $\bar{r}(t) = 1$ if $t > r$. For $u, v \in \mathbf{R}_\phi^+$, we define $u \preceq v$ iff $v(t) \leq u(t)$ for all $t \in \mathbf{R}$. If $\mathcal{A}$ is a non-empty subset of $\mathbf{R}_\phi^+$ and if $u_o \in \mathbf{R}_\phi^+$ is defined by $u_o(t) = \sup_{v \in \mathcal{A}} v(t)$, then u_o is the biggest of all $u \in \mathbf{R}_\phi^+$ with $u \preceq v$ for all $v \in \mathcal{A}$. We will denote u_o by $\inf \mathcal{A}$ or by $\bigwedge \mathcal{A}$. For $u_1, u_2 \in \mathbf{R}_\phi^+$, we define $u = u_1 \oplus u_2 \in \mathbf{R}_\phi^+$ by $u(t) = \sup\{u_1(t_1) \wedge u_2(t_2) : t = t_1 + t_2\}$. Also, for $u \in \mathbf{R}_\phi^+$ and $\lambda > 0$, we define λu by $(\lambda u)(t) = u(\lambda^{-1} t)$. It is easy to see that, for $u \in \mathbf{R}_\phi^+$ and $\lambda > 0$, we have $(\bar{\lambda} \oplus u)(t) = u(t - \lambda)$.

Definition 2.1 *A probabilistic pseudometric on a set X (see [1]) is a mapping $F : X \times X \to \mathbf{R}_\phi^+$ such that, for all x, y, z in X, we have*

$$F(x,x) = \bar{0}, F(x,y) = F(y,x), F(x,z) \preceq F(x,y) \oplus F(y,z).$$

If in addition $F(x,y)(0+) = 0$ when $x \neq y$, then F is called a probabilistic metric.

If r_1, r_2 are non-negative real numbers, then $\overline{r_1} \preceq \overline{r_2}$ iff $r_1 \leq r_2$. Also, for $r = |r_1 - r_2|$, we have that

$$\bar{r} = \wedge\{u \in \mathbf{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1} \text{ and } \overline{r_1} \preceq u \oplus \bar{r_2}\}.$$

In fact, let $u_o = \wedge\{u \in \mathbf{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1}$ and $\overline{r_1} \preceq u \oplus \overline{r_2}\}$ and assume (say) $r_1 \geq r_2$. Let $u \in \mathcal{R}_\phi^+$ be such that $\bar{r_2} \preceq u \oplus \overline{r_1}, \overline{r_1} \preceq u \oplus \overline{r_2}$. Then $\overline{r_1}(t) \geq (u \oplus \overline{r_2}(t) = u(t - r_2)$ for all t. If $s < r_1$, then $0 = \overline{r_1}(s) \geq u(s - r_2)$ and so $u(r_1 - r_2) = \sup_{s < r_1} u(s - r_2) = 0$ which implies that $\bar{r} \preceq u$. Thus $\bar{r} \preceq u_o$. On the other hand, we have $\bar{r} \oplus \overline{r_2} = \overline{r_1}$ and $\bar{r} \oplus \overline{r_1} = \overline{2r_1 - r_2}$. Since $\overline{r_2} \preceq \overline{2r_1 - r_2}$, it follows that $u_o \preceq \bar{r}$ and hence $\bar{r} = u_o$. Motivated from the above we define the following distance function on $\mathcal{R}_\phi^+$

$$D : \mathcal{R}_\phi^+ \times \mathcal{R}_\phi^+ \longrightarrow \mathcal{R}_\phi^+, \quad D(u_1, u_2) = \wedge\{u \in \mathcal{R}_\phi^+ : u_1 \preceq u_2 \oplus u, u_2 \preceq u \oplus u_1\}.$$

Then D is a probabilistic pseudometric on $\mathcal{R}_\phi^+$. In fact, it is clear that $D(u_1, u_2) = D(u_2, u_1)$. Also, since $u = u \oplus \bar{0}$, when $u \in \mathcal{R}_\phi^+$, we have that $D(u, u) = \bar{0}$. Finally, let $D(u_1, u_2)(t_1) \wedge D(u_2, u_3)(t_2) > \theta > 0$. There are $v_1, v_2 \in \mathcal{R}_\phi^+$ with

$u_1 \preceq v_1 \oplus u_2, u_2 \preceq v_1 \oplus u_1, u_3 \preceq v_2 \oplus u_2, u_2 \preceq v_2 \oplus u_3, v_1(t_1) > \theta, v_2(t_2) > \theta$. Now $u_1 \preceq v_1 \oplus u_2 \preceq v_1 \oplus (v_2 \oplus u_3) = (v_1 \oplus v_2) \oplus u_3$ and $u_3 \preceq v_2 \oplus u_2 \preceq v_2 \oplus (v_1 \oplus u_1) = (v_1 \oplus v_2) \oplus u_1$.Thus $D(u_1, u_3) \preceq v_1 \oplus v_2$ and $D(u_1, u_3)(t_1 + t_2) \geq v_1(t_1) \wedge v_2(t_2) > \theta$. This proves that $D(u_1, u_3) \preceq D(u_1, u_2) \oplus D(u_2, u_3)$ and the claim follows. We will refer to D as the usual probabilistic pseudometric on $\mathcal{R}_\phi^+$.
Let now F be a probabilistic psedometric on X. For $t > 0$, let $u_{F,t}$ be defined on X^2 by $u_{F,t}(x, y) = F(x, y)(t)$. The family $\mathcal{B}_F = \{u_{F,t} : t > 0\}$ is a base for a fuzzy uniformity $\mathcal{U}_F$ on X. Let τ_F be the fuzzifying topology induced by $\mathcal{U}_F$.
In the rest of the paper, we will consider on $\mathcal{R}_\phi^+$ the fuzzifying topology induced by the usual probabilistic pseudometric D.

Theorem 2.2 *A probabilistic pseudometric F, on a fuzzifying topological space (X, τ), is $\tau \times \tau$ continuous iff $\tau_F \leq \tau$.*

Proof : Assume that $\tau_F \leq \tau$ and let G be a subset of $\mathbf{R}_\phi^+$ and $u = F(x_o, y_o)$ with $N_u(G) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{v \notin G} D(v, u)(t) > \theta$. For x, y in X, we have

$$F(x, y) \preceq F(x, x_o) \oplus F(x_o, y_o) \oplus F(y_o, y) = [F(x, x_o) \oplus F(y, y_o)] \oplus F(x_o, y_o).$$

Similarly $F(x_o, y_o) \preceq [F(x, x_o) \oplus F(y, y_o)] \oplus F(x, y)$. Thus

$$D(F(x, y), F(x_o, y_o)) \preceq F(x, x_o) \oplus F(y, y_o).$$

Let

$$A_1 = \{x \in X : F(x, x_o)(t/2) \geq 1 - \theta\}, \quad \text{and} \quad A_2 = \{x \in X : F(y, y_o)(t/2) \geq 1 - \theta\}.$$

If $x \in A_1, y \in A_2$, then

$$D(F(x, y), F(x_o, y_o))(t) \geq F(x, x_o)(t/2) \wedge F(y, y_o)(t/2) \geq 1 - \theta$$

and so $F(x, y) \in G$. Also, $N_{x_o}^\tau(A_1) \geq N_{x_o}^{\tau_F}(A_1) \geq 1 - \sup_{x \notin A_1} F(x, x_o)(t/2) \geq \theta$ and $N_{y_o}^\tau(A_2) \geq \theta$. Therefore,

$$N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{x_o}^\tau(A_1) \wedge N_{y_o}^\tau(A_1) \geq \theta,$$

which proves that $N_{(x_o, y_o)}^{\tau \times \tau}(F^{-1}(G)) \geq N_{f(x_o, y_o)}(G)$ and so F is $\tau \times \tau$ continuous.
Conversely, assume that F is $\tau \times \tau$ continuous and let $N_{x_o}^{\tau_F}(A) > \theta > 0$. Choose $\epsilon > 0$ such that $N_{x_o}^{\tau_F}(A) > \theta + \epsilon$. There exists a $t > 0$ such that $1 - \sup_{x \notin A} F(x, x_o)(t) > \theta + \epsilon$. If

$$Z = \{u \in \mathbf{R}_\phi^+ : D(u, \bar{0})(t) = u(t) > 1 - \theta - \epsilon\},$$

then

$$N_{\bar{0}}(Z) \geq 1 - \sup_{u \notin Z} D(u, \bar{0})(t) \geq \theta + \epsilon > \theta.$$

Since F is $\tau \times \tau$ continuous and $F(x_o, x_o) = \bar{0}$, there exists a subset A_1 of X containing x_o such that $A_1 \times A_1 \subset F^{-1}(Z)$ and $N_{x_o}(A_1) > \theta$. If $x \in A_1$, then $F(x, x_o) \in Z$ and so $F(x, x_o)(t) > 1 - \theta - \epsilon$, which implies that $x \in A$. Thus $A_1 \subset A$ and so $N_{x_o}(A) \geq N_{x_o}^{\tau_F}(A)$ for every subset A of X and every $x_o \in X$. Hence $\tau_F \leq \tau$ and the result follows.

Theorem 2.3 *Let F be a probabilistic pseudometric on a set X, $\tau = \tau_F, (x_\delta)_{\delta\in\Delta}$ a net in X and $x \in X$. Then*

$$c(x_\delta \to x) = \inf_{t>0} \liminf_{\delta} F(x_\delta, x)(t).$$

Proof: Let $d = \inf_{t>0} \liminf_\delta F(x_\delta, x)(t)$ and assume that $d < \theta < 1$. There exists a $t > 0$ such that $\liminf_\delta F(x_\delta, x)(t) < \theta$. Let $A = \{y : F(y,x)(t) > \theta\}$. Then (x_δ) is not eventually in A and so $c(x_\delta \to x) \leq 1 - N_x(A) \leq \sup_{y\notin A} F(y,x)(t) \leq \theta$, which proves that $c(x_\delta \to x) \leq d$. On the other hand, let $c(x_\delta \to x) < r < 1$. There exists a subset B of X such that (x_δ) is not eventually in B and $1 - N_x(B) < r$. Let $s > 0$ be such that $1 - \sup_{y\notin B} F(y,x)(s) > 1 - r$. For each $\delta \in \Delta$, there exists $\delta' \geq \delta$ with $x_{\delta'} \notin B$ and so $F(x_{\delta'}, x)(s) \leq \sup_{y\notin B} F(y,x)(s)$. Thus $d \leq \liminf_\delta F(x_\delta, x)(s) < r$, which proves that $d \leq c(x_\delta \to x)$ and the result follows.

Theorem 2.4 *Let $F_1, F_2, \ldots, F_n$ be probabilistic pseudometrics on X and define F by*

$$F(x,y)(t) = \min_{1\leq k\leq n} F_k(x,y)(t).$$

Then F is a probabilistic pseudometric and $\tau_F = \bigvee_{k+1}^{n} \tau_{F_k}$.

Proof: Using induction on n, it suffices to prove the result in the case of $n = 2$. It follows easily that F is a probabilistic pseudometric. Since $F_1, F_2 \preceq F$, it follows that $\tau_{F_1}, \tau_{F_2} \leq \tau_F$ and so $\tau_o = \tau_{F_1} \vee \tau_{F_2} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{y\notin A} F(y,x)(t) > \theta$. Let $B_i = \{y \in A^c : F_i(y,x)(t) < 1 - \theta\}, i = 1, 2$. Then $A^c = B_1 \cup B_2$ and so $A = A_1 \cap A_2, A_i = B_i^c$. Moreover $N_x^{\tau_{F_i}}(A_i) \geq 1 - \sup_{y\in B_i} F_i(y,x)(t) \geq \theta$ and thus

$$N_x^{\tau_o}(A) \geq N_x^{\tau_o}(A_1) \bigwedge N_x^{\tau_o}(A_2) \geq N_x^{\tau_{F_1}}(A_1) \bigwedge N_x^{\tau_{F_2}}(A_2) \geq \theta$$

This proves that $N_x^{\tau_o}(A) \geq N_x^{\tau_F}(A)$ and the result follows.

For $\mathcal{F}$ a family of probabilistic pseudometrics on a set X, we will denote by $\tau_{\mathcal{F}}$ the supremum of the fuzzifying topologies $\tau_F, F \in \mathcal{F}$, i.e. $\tau_{\mathcal{F}} = \bigvee_{F\in\mathcal{F}} \tau_F$.

Theorem 2.5 *If $\tau = \tau_{\mathcal{F}}$, where $\mathcal{F}$ is a family of probabilistic pseudometrics on a set X, then $T_2(X) = T_1(X) = 1 - \sup_{y\neq x} \inf_{F\in\mathcal{F}} F(x,y)(0+)$.*

Proof: Let $d = 1 - \sup_{y\neq x} \inf_{F\in\mathcal{F}} F(x,y)(0+)$. It is always true that $T_2(X) \leq T_1(X)$. Suppose that $T_1(X) > r > 0$ and let $x \neq y$. Since τ^r is T_1, there exists a τ^r-neighborhood A of x not containing y. Now $N_x(A) > r$ and hence, there are subsets $A_1, \ldots, A_n$ of X and $F_1, \ldots, F_n$ in $\mathcal{F}$ such that $\bigcap A_k \subset A, N_x^{\tau_{F_k}}(A_k > r$. Since y is not in A, there exists a k with $y \notin A_k$. Let $t > 0$ be such that

$$1 - \sup_{z\notin A_k} F_k(z,x)(t) > r \quad \text{and so} \quad \inf_{F\in\mathcal{F}} F(x,y)(t)(0+) \leq F_k(x,y)(t) < 1 - r,$$

which proves that $d \geq r$. Thus $d \geq T_1(X)$. On the other hand, assume that $d > \theta > 0$ and let $x \neq y$. Choose $\epsilon > 0$ such that $d > \theta + \epsilon$. There exists $F \in \mathcal{F}$ with $F(x,y)(0+) < 1 - \theta - \epsilon$ and hence $F(x,y)(t) < 1 - \theta - \epsilon$ for some $t > 0$. Let

$$A = \{z : F(z,x)(t/2) > 1 - \theta - \epsilon\}, \quad B = \{z : F(z,y)(t/2) > 1 - \theta - \epsilon\}.$$

Clearly $x \in A, y \in B$. If $z \in A \cap B$, then

$$F(x,y)(t) \geq F(x,z)(t/2) \wedge F(z,y)(t/2) > 1 - \theta - \epsilon,$$

a contradiction. Thus $A \cap B = \emptyset$. Moreover

$$N_x(A) \geq N_x^{\tau_F}(A) \geq 1 - \sup_{z \notin A} F(x,z)(t/2) \geq \theta + \epsilon > \theta \text{ and } N_y(A) > \theta.$$

It follows that $T_2(X) \geq d$ and the proof is complete.

Let us say that a fuzzifying topology τ on a set X is pseudometrizable if there exists a probabilistic pseudometric F on X with $\tau = \tau_F$.

Theorem 2.6 *A fuzzifying topology τ on X is pseudometrizable iff each level topology $\tau^\theta, 0 \leq \theta < 1$, is pseudometrizable.*

Proof: Assume that $\tau = \tau_F$ for some probabilistic pseudometric F and let $0 \leq \theta < 1$. For each positive integer$n$, with $n > 1/(1-\theta)$, let

$$A_n = \{(x,y) \in X^2 : F(x,y)(1/n) > 1 - \theta - 1/n\}.$$

Then $A_{n+1} \subset A_n$ and the family $\mathcal{D} = \{A_n : n \in \mathbf{N}, n > 1/(1-\theta)\}$ is a base for a uniformity $\mathcal{U}$ on X. The topology σ_θ induced by $\mathcal{U}$ is pseudometrizable since $\mathcal{D}$ is countable. Moreover $\sigma_\theta = \tau^\theta$. Indeed, let A be a σ_θ-neighborhood of x. There exists $n \in \mathbf{N}, n > 1/(1-\theta)$, such that $B = \{y : F(x,y)(1/n) > 1 - \theta - 1/n\} \subset A$. Now

$$N_x^\tau(A) \geq N_x^\tau(B) \geq 1 - \sup_{y \notin B} F(x,y)(1/n) \geq \theta + 1/n > \theta$$

and so A is a τ^θ-neighborhood of x. Conversely, assume that A is a τ^θ-neighborhood of x. There exists $\epsilon > 0$ with $N_x(A) > \theta + \epsilon$. Now there exists a positive integer $n > 1/\epsilon$ such that $1 - \sup_{y \notin A} F(x,y)(1/n) > \theta + 1/n$. Hence

$$\{y : F(x,y)(1/n) > 1 - \theta - 1/n\} \subset A,$$

which implies that A is a σ_θ-neighborhood of x. Thus $\tau^\theta = \sigma_\theta$ and therefore each τ^θ is pseudometrizable. Conversely, suppose that each τ^θ is pseudometrizable. By an argument analogous to the one used in the proof of Theorem 3.3 in [4], we show that there exists a family $\{d_\theta : 0 \leq \theta < 1\}$ of pseudometrics on X such that $d_\theta = \sup_{\theta_1 > \theta} d_{\theta_1}$, for each $0 \leq \theta < 1$, and τ^θ coincides with the topology induced by the pseudometric d_θ. Now, for x, y in X, define $F(x,y) : \mathbf{R} \to [0,1]$ by $F(x,y)(t) = 0$ if $t \leq 0$ and $F(x,y)(t) = \sup\{\theta : 0 < \theta \leq 1, d_{1-\theta}(x,y) < t\}$ if $t > 0$. It is clear $F(x,y)$ is increasing and left continuous. For $0 < r < 1$ and $t > d_{1-r}(x,y)$, we have that $F(x,y)(t) \geq r$ and so $\lim_{t\to\infty} F(x,y)(t) = 1$. Also $F(x,x)(t) = 1$ for every x and every $t > 0$. To show that F is a probabilistic pseudometric on X, we must prove that it satisfies the triangle inequality. So, let $F(x,y)(t_1) \wedge F(y,z)(t_2) > \theta > 0$. Then $d_{1-\theta}(x,y) < t_1, d_{1-\theta}(y,z) < t_2$ and so $d_{1-\theta}(x,z) < t_1 + t_2$, which implies that $F(x,z)(t_1 + t_2) \geq \theta$. Thus the triangle inequality is satisfied and F is a probabilistic pseudometric. We will finish the proof by showing that $\tau_F = \tau$. So let $N_x^{\tau_F} > \theta > 0$

and choose $t > 0$ such that $1 - \sup_{y \notin A} F(y,x)(t) > \theta$. If now $d_\theta(x,y) < t$, then $F(x,y)(t) \geq 1-\theta$ and thus $y \in A$, which proves that A is a $\sigma_\theta = \tau^\theta$ neighborhood of x. Hence $\tau \geq \tau_F$. On the other hand, let B be a τ^θ-neighborhood of x. There exists $\theta_1 > \theta$ such that $N_x(B) > \theta_1$. Now B is a τ_{θ_1}-neighborhood of x and so there exists $t > 0$ such that $\{y : d_{\theta_1}(x,y) < t\} \subset B$. If $F(x,y)(t) > 1-\theta_1$, then there exists $\alpha > 1-\theta_1$ such that $d_{1-\alpha}(x,y) < t$ and so $d_{\theta_1}(x,y) < t$. Thus $\{y : F(x,y)(t) > 1-\theta_1\} \subset B$ and therefore

$$N_x^{\tau_F}(B) \geq 1 - \sup_{y \notin B} F(x,y)(t) \geq \theta_1 > \theta.$$

Thus $\tau_F \geq \tau$ and the result follows.

Theorem 2.7 *Let (X,F) be a probabilistic pseudometric space, $A \subset X$ and $x \in X$. Let*

$$\begin{aligned} \alpha &= \sup\{\inf_{t>0} \liminf_n F(x_n,x)(t) : (x_n) \text{ sequence in } A\} \\ \beta &= \sup\{\liminf_n F(x_n,x)(t_n) : t_n \to 0+, (x_n) \text{ sequence in } A\} \\ \gamma &= \sup\{\liminf_n F(x_n,x)(1/n) : (x_n) \text{ sequence in } A\} \end{aligned}$$

Then $\alpha = \beta = \gamma = \bar{A}(x)$.

Proof: If $(x_n) \subset A$, then

$$\bar{A}(x) \geq c(x_n \to x) = \inf_{t>0} \liminf_n F(x_n,x)(t)$$

and so $\bar{A}(x) \geq \alpha$. Assume that $\beta > \theta > 0$. There exist a sequence (x_n) in A and a sequence (t_n) of positive real numbers, with $t_n \to 0+$, such that $\liminf_n F(x_n,x)(t_n) > \theta$. Let $t > 0$ and choose k such that $t_n < t$ when $n \geq k$. For $m \geq k$ we have $\inf_{n>m} F(x_n,x)(t) \geq \inf_{n \geq m} F(x_n,x)(t_n) > \theta$. Thus $\liminf_n F(x_n,x)(t) > \theta$ for each $t > 0$ and so $\alpha \geq \theta$, which proves that $\alpha \geq \beta$. Clearly $\beta \geq \gamma$. Finally, $N_x(A^c) \geq 1 - sup_{y \in A} F(y,x)(1/n)$ and so $\sup_{y \in A} F(y,x)(1/n) \geq 1 - N_x(A^c) = \bar{A}(x) > \bar{A}(x) - 1/n$. Hence, for each $n \in \mathbf{N}$, there exists $x_n \in A$ with $F(x_n,x)(1/n) > \bar{A}(x) - 1/n$. Consequently,

$$\gamma \geq \liminf_n F(x_n,x)(1/n) \geq \liminf_n (\bar{A}(x) - 1/n) = \bar{A}(x)$$

and so $\gamma \geq \bar{A}(x) \geq \alpha \geq \beta \geq \gamma$, which completes the proof.

In view of [4], Theorem 4.14, we have the following

Corollary 2.8 *Every pseudometrizable fuzzifying topological space is* **N***-sequential and hence sequential.*

Theorem 2.9 *If (F_n) is a sequence of probabilistic pseudometrics on a set X, then there exists a probabilistic pseudometric F such that $\tau_F = \bigvee_n \tau_{F_n}$.*

Proof: If F is a probabilistic pseudometric on X and if $\bar{F}$ is defined by $\bar{F}(x,y)(t) = F(x,y)(t)$ if $t \leq 1$ and $\bar{F}(x,y)(t) = 1$ if $t > 1$, then $\bar{F}$ is a probabilistic pseudometric on X and $\tau_{\bar{F}} = \tau_F$. Hence, we may assume that $F_n(x,y)(t) = 1$, for all n, if $t > 1$.

For x, y in X, define $F(x,y)$ on $\mathbf{R}$ by $F(x,y)(t) = 0$ if $t \leq 0$ and $F(x,y)(t) = \inf_n[\frac{1}{n}F_n(x,y)](t)$ if $t > 0$. Clearly $F(x,y)$ is increasing and $F(x,y)(t) = 1$ if $t > 1$. Also $F(x,y)$ is left continuous. In fact, let $F(x,y)(t) > \theta > 0$ and choose n such that $(n+1)t > 1$. There exists $0 < s_1 < t$ such that $F_k(x,y)(ks_1) > \theta$ for $k = 1, \ldots, n$. Choose $s_1 < s < t$ such that $(n+1)s > 1$. Now $F_m(x,y)(ms) = 1$ if $m > n$. Thus

$$F(x,y)(s) = \min_{1\leq k\leq n}[\frac{1}{k}F_k(x,y)](s) > \theta,$$

which proves that $F(x,y)$ is in $\mathbf{R}_\phi^+$. It is clear that $F(x,x) = \bar{0}$. We need to prove that F satisfies the triangle inequality. So assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > \theta > 0$. If m is such that $(m+1)(t_1+t_2) > 1$, then

$$F(x,z)(t_1+t_2) = \min_{1\leq k\leq m} F_k(x,z)(k(t_1+t_2)).$$

Since

$$F_k(x,z)(k(t_1+t_2)) \geq F_k(x,y)(kt_1) \wedge F_k(y,z)(kt_2) > \theta,$$

it follows that $F(x,z)(t_1+t_2) > \theta$ and so F satisfies the triangle inequality. We will finish the proof by showing that $\tau_F = \bigvee \tau_{F_n}$. To see this, we first observe that $\frac{1}{n}F_n \preceq F$ which implies that $\tau_{F_n} = \tau_{\frac{1}{n}F_n} \leq \tau_F$ and so $\tau_o = \bigvee_n \tau_{\frac{1}{n}F_n} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta$ and choose $\epsilon > 0$ such that $N_x^{\tau_F}(A) > \theta + \epsilon$. Let $t > 0$ be such that $1 - \sup_{y\notin A} F(y,x)(t) > \theta + \epsilon$. If $(m+1)t > 1$, then

$$F(y,z)(t) = \min_{1\leq k\leq m} F_k(y,z)(kt).$$

Let $A_k = \{y : F_k(y,x)(kt) \geq 1 - \theta - \epsilon\}$. Then

$$N_x^{\tau_o}(A_k) \geq N_x^{\tau_{F_k}}(A_k) \geq 1 - sup_{z\notin A_k} F_k(z,x)(kt) \geq \theta + \epsilon > \theta$$

and $\bigcap_{k=1}^m A_k \subset A$. Hence $N_x^{\tau_o}(A) \geq \min_{1\leq k\leq m} N_x^{\tau_o}(A_k) > \theta$. This proves that $\tau_F \leq \tau_o$ and the result follows.

Theorem 2.10 *Let $f : X \to Y$ be a function and let F be a probabilistic pseudometric on Y. Then the function*

$$f^{-1}(F) : X^2 \to \mathbf{R}_\phi^{+1}, f^{-1}(F)(x,y) = F(f(x), f(y))$$

is a probabilistic pseudometric on X and $\tau_{f^{-1}(F)} = f^{-1}(\tau_F)$.

Proof: It follows easily that $f^1(F)$ is a probabilistic pseudometric on X. Let $x \in X$ and $B \subset X$. If $D = Y \setminus f(B^c)$, then

$$\begin{aligned} N_x^{\tau_{f^{-1}(F)}}(B) &= \inf_{t>0}[1 - \sup_{y\notin B} F(f(y), f(x))(t)] \\ &= \inf_{t>0}[1 - \sup_{z\in D^c} F(z, f(x))(t)] \\ &= N_{f(x)}^{\tau_F}(D) = N_x^{f^{-1}(\tau_F)}(B), \end{aligned}$$

which clearly completes the proof.

Corollary 2.11 *If F is a probabilistic pseudometric on a set X and $Y \subset X$, then $\tau_F|_Y$ is induced by the probabilistic pseudometric $G = F|_{Y\times Y}, G(x,y) = F(x,y)$.*

Corollary 2.12 *If (X_n, τ_n) is a sequence of pseudometrizable fuzzifying topological spaces, then the cartesian product $(X, \tau) = (\prod X_n, \prod \tau_n)$ is pseudometrizable.*

Proof: Let F_n be be a probabilistic pseudometric pseudometric on X_n inducing τ_n. If $G_n = \pi_n^{-1}(F_n)$, then $\tau_{G_n} = \pi_n^{-1}(\tau_n)$ and so $\tau = \bigvee_n \pi_n^{-1}(\tau_n)$ is pseudometrizable.

3 Level Proximities

Let δ be a fuzzifying proximity on a set X. For each $0 < d \leq 1$, let δ^d be the binary relation on 2^X defined by : $A\delta^d B$ iff $\delta(A,B) \geq d$. It is easy to see that δ^d is a classical proximity on X. We will show that the classical topology σ_d induced by δ^d coincides with τ^{1-d}. In fact, let $x \in A \in \sigma_d$. Then, x is not in the σ_d-closure of A^c, which implies that $x \not\delta^d A^c$, i.e. $\delta(x, A^c) < d$, and so $N_x^\tau(A) = 1 - \delta(x, A^c) > 1 - d$. This proves that $A \in \tau^{1-d}$. Conversely, if $x \in B \in \tau^{1-d}$, then $N_x^\tau(A) > 1 - d$ and thus $\delta(x, A^c) < d$, which implies that x is not in the σ_d-closure of B^c. Hence B^c is σ_d-closed and so B is σ_d-open.

Theorem 3.1 *If δ is a fuzzifying proximity on a set X and $0 < d \leq 1$, then*

$$\delta^d = \bigvee_{0<\theta<d} \delta^\theta.$$

Proof: If $0 < \theta < d$, then δ^θ is coarser than δ^d and so $\delta_o = \bigvee_{0<\theta<d} \delta^\theta$ is coarser than δ^d. On the other hand, let $A\delta_o B$. Since δ_o is finer than δ^θ (for $0 < \theta < d$), we have that $A\delta^\theta B$ and so $\delta(A,B) \geq \theta$, for each $0 < \theta < d$, which implies that $\delta(A,B) \geq d$, i.e. $A\delta^d B$. So δ_o is finer than δ^d and the result follows.

Theorem 3.2 *For a family $\{\gamma_d : 0 < d \leq 1\}$ of classical proximities on a set X the following are equivalent:*

(1) There exists a fuzzifying proximity δ on X such that $\delta^d = \gamma_d$ for all d.

(2) $\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$ for each $0 < d \leq 1$.

Proof: In view of the preceding Theorem, (1) implies (2). Assume now that (2) is satisfied and define δ on $2^X \times 2^X$ by $\delta(A,B) = \sup\{d : A\gamma_d B\}$ (the supremum over the empty family is taken to be zero). It is clear that $\delta(A,B) = 1$ if the A, B are not disjoint. Also $\delta(A,B) = \delta(A,B)$ and $\delta(A,B \geq \delta(A_1,B_1)$ if $A_1 \subset A, B_1 \subset B$. Let now $\delta(A,B) < d < 1$. Then $A \not\gamma_d B$ and so there exists a subset D of X such that $A \not\gamma_d D$ and $D^c \not\gamma_d B$. Since $A \not\gamma_d D$, we have that $\delta(A,D) \leq d$. Similarly $\delta(D^c, B) \leq d$ and so $\inf\{\delta(A,D) \wedge \delta(D^c,B)\} \leq \delta(A,B)$. On the other hand, if $\delta(A,D) \wedge \delta(D^c,B) < \theta < 1$, then $A \subset D^c$ and so $\delta(A,B) \leq \delta(D^c,B) < \theta$. This proves that δ is a fuzzifying proximity on X. We will finish the proof by showing that $\delta^d = \gamma_d$ for all d. Indeed, if $A\gamma_d B$, then $\delta(A,B) \geq d$, i.e. $A\delta^d B$. On the other hand, let $A\delta^d B$ and let $(A_i), (B_j)$ be finite families of subsets of X with $A = \cup_i, B = \cup B_j$. Since $\delta(A, B) = \bigvee_{i,j} \delta(A_i, B_j) \geq d$, there exists a pair (i, j) such that $\delta(A_i, B_j) \geq d$. If now $0 < \theta < d$, then there exists $r > \theta$ with $A_i \gamma_r B_j$ and so $A_i \gamma_\theta B_j$. This proves that $A\gamma_d B$ since $\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$. This completes the proof.

Theorem 3.3 *Let $(X, \delta_1), (Y\delta_2)$ be fuzzifying proximity spaces and let $f : X \to Y$ be a function. Then f is proximally continuous iff $f : (X, \delta_1^d) \to (Y, \delta_2^d)$ is proximally continuous for each $0 < d \leq 1$.*

Proof: It follows immediately from the definitions.

Theorem 3.4 *Let $(X_\lambda, \delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximity spaces and let $(X, \delta) = (\prod X_\lambda, \prod \delta_\lambda)$ be the product fuzzifying proximity space. Then $\delta^d = \prod \delta_\lambda^d$ for all $0 < d \leq 1$.*

Proof: Since each projection $\pi_\lambda : (X, \delta^d) \to (X_\lambda, \delta_\lambda^d)$ is proximally continuous, it follows that δ^d is finer than $\sigma = \prod \delta_\lambda^d$. On the other hand, let $A\sigma B$. We need to show that $\delta(A, B) \geq d$. In fact, let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \cup A_i, B = \cup B_j$. Since $A\sigma B$ and $\sigma = \bigvee_\lambda \pi_\lambda^{-1}(\delta_\lambda^d)$, there exists a pair (i, j) such that $A_i \pi_\lambda^{-1}(\delta^d) B_j$, i.e. $\delta_\lambda(\pi_\lambda(A_i), \pi_\lambda(B_j)) \geq d$. In view of Theorem 8.9 in [2], we conclude that $\delta(A, B) \geq d$. Hence $\sigma = \delta^d$ and the proof is complete.

We have the following easily established

Theorem 3.5 *Let (Y, δ) be a fuzzifying proximity space and let $f : X \to Y$. Then $f^{-1}(\delta)^d = f^{-1}(\delta^d)$ for each $0 < d \leq 1$.*

Theorem 3.6 *Let $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X and $\delta = \vee_\lambda \delta_\lambda$. Then $\delta^d = \bigvee_\lambda \delta_\lambda^d$ for each $0 < d \leq 1$.*

Proof: Let $\sigma = \bigvee_\lambda \delta_\lambda^d$. Since δ is finer than each δ_λ, it follows that δ^d is finer than each δ_λ^d and so δ^d is finer than σ. On the other hand, let $A\sigma B$ and let $(A_i), (B_j)$ be finite families of subsets of X such that $A = \cup A_i, B = \cup B_j$. There exists a pair (i, j) such that $A_i \sigma B_j$. Since σ is finer than each δ_λ^d, we have that $A_i \delta_\lambda^d B_j$, i.e. $\delta_\lambda(A_i, B_j) \geq d$. In view of Theorem 8.10 in [2], we get that $\delta(A, B) \geq d$, i.e. $A\delta^d B$. So σ is finer than δ^d and the proof is complete.

4 Completely Regular Fuzzifying Spaces

Definition 4.1 *A fuzzifying topological space (X, τ) is called completely regular if each of the classical level topologies $\tau^d, 0 \leq d < 1$ is completely regular.*

Definition 4.2 *A fuzzifying proximity δ on a set X is said to be compatible with a fuzzifying topology τ if τ coincides with the fuzzifying topology τ_δ induced by δ.*

We have the following easily established

Theorem 4.3 *Subspaces and cartesian products of completely regular fuzzifying spaces are completely regular.*

Theorem 4.4 *Let (X, τ) be a completely regular fuzzifying topological space and define $\delta = \delta(\tau) : 2^X \times 2^X \to [0,1]$ by*

$$\delta(A,B) = 1 - \sup\{d : 0 \leq d < 1, \exists f : (X, \tau^d) \to [0,1] \text{ continuous } f(A) = 0, f(B) = 1\}.$$

Then: (1) δ is a fuzzifying proximity on X compatible with τ.

(2) If δ_1 is any fuzzifying proximity on X compatible with τ, then δ is finer than δ_1.

Proof: It is easy to see that δ satisfies $(FP1), (FP2), (FP3)$ and $(FP5)$. We will prove that δ satisfies $(FP4)$. Let

$$\alpha = \inf\{\delta A, D) \vee \delta(D^c, B) : D \subset X\}.$$

If $\delta(A, D) \vee \delta(D^c, B) < \theta$, then $A \subset D^c$ and so $\delta(A, B) \leq \delta(D^c, B) < \theta$, which proves that $\delta(A, B) \leq \alpha$. On the other hand, assume that $\delta(A, B) < r < 1$. There exist a $d, 1 - r < d < 1$, and $f : X \to [0,1]$ τ^d-continuous such that $f(A) = 0, f(B) = 1$. Let $D = \{x \in X : 1/2 \leq f(x) \leq 1\}$ and define $h_1, h_2 : [0,1] \to [0,1], h_1(t) = 2t, h_2(t) = 0$ if $0 \leq t \leq 1/2$ and $h_1(t) = 1, h_2(t) = 2t - 1$ if $1/2 < t \leq 1$. If $g_i = h_i \circ f, i = 1, 2$, then $g_1(A) = 0, g_1(D) = 1, g_2(D^c) = 0, g_2(B) = 1$. Thus $\delta(A, D) \leq 1 - d < r, \delta(D^c, B) < r$, which proves that $\alpha \leq \delta(A, B)$. Hence δ is a fuzzifying proximity on X. We need to show that $\tau = \tau_\delta$. So, let $\tau(A) > \theta > 0$. Since τ^θ is completely regular, given $x \in A$, there exists $f_x : X \to [0,1], \tau^\theta$-continuous, $f_x(x) = 0, f_x(A^c) = 1$. Thus $\delta(x, A^c) \leq 1 - \theta$ and so $N_x^{\tau_\delta}(A) = 1 - \delta(x, A^c) \geq \theta$. It follows that $\tau_\delta(A) = \inf_{x \in A} N_x^{\tau_\delta}(A) \geq \theta$, which proves that $\tau_\delta \geq \tau$. On the other hand, assume that $\tau_\delta(A) > r > 0$. If $x \in A$, then $\delta(x, A^c) = 1 - N_x^{\tau_\delta}(A) < 1 - r$, and therefore there exists a $d, 0 < 1 - d < 1 - r$ and $f : X \to [0,1]$ τ^d-continuous such that $f(x) = 0, f(A^c) = 1$. The set $G = \{y : f(y) < 1/2\}$ is in τ^d and $x \in G \subset A$. Thus

$$N_x^\tau(A) \geq N_x^\tau(G) \geq d > r.$$

This proves that $\tau(A) \geq r$ and so $\tau \geq \tau_\delta$, which completes the proof of (1).
Let δ_1 be a fuzzifying proximity on X compatible with τ and let A, B be subsets of X with $\delta_1(A, B) < \theta < 1$. If $d = 1 - \theta$, then δ_1^θ is compatible with τ^d. Since $A \not\delta_1^\theta B$, there exists (by [11], Remarks 3.15) an $f : X \to [0,1]$ τ_d-continuous, with $f(A) = 0, f(B) = 1$, and so $\delta(A, B) \leq 1 - d = \theta$, which proves that $\delta(A, B) \leq \delta_1(A, B)$ and therefore δ is finer than δ_1. This completes the proof.

Theorem 4.5 *For a fuzzifying topological space (X, τ), the following are equivalent:*
(1) (X, τ) is completely regular.
(2) There exists a fuzzifying proximity δ on X compatible with τ.
(3) (X, τ) is fuzzy uniformizable, i.e. there exists a fuzzy uniformity $\mathcal{U}$ on X such that τ coincides with the fuzzifying topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$.

Proof: By [3], (2) is equivalent (3). Also (1) implies (2) in view of the preceding Theorem. Assume now that $\tau = \tau_\delta$ for some fuzzifying proximity δ. For each $0 < d \leq 1, \delta^d$ is a classical proximity compatible with τ^{1-d} and so τ^{1-d} is completely regular. This completes the proof.

Theorem 4.6 *Every pseudometrizable fuzzy topological space (X, τ) is completely regular.*

Proof: If τ is pseudometrizable, then each $\tau^d, 0 \leq d < 1$, is pseudometrizable and hence τ^d is completely regular.

Theorem 4.7 *For a fuzzifying topological space (X, τ), the following are equivalent:*
(1) (X, τ) is completely regular.
(2) If $\mathcal{F} = \mathcal{F}_\tau = cp(X)$ is the family of all probabilistic pseudometrics on X which are $\tau \times \tau$ continuous as functions from X^2 to $\mathbf{R}_\phi^+$, then $\tau = \tau_{\mathcal{F}_\tau}$.
(3) There exists a family $\mathcal{F}$ of probabilistic pseudometrics on X such that $\tau = \tau_{\mathcal{F}}$.

Proof: (1) $\Rightarrow$ (2). For each $F \in \mathcal{F}_\tau$, we have that $\tau_F \leq \tau$ (by Theorem 2.2) and so $\tau_{\mathcal{F}_\tau} \leq \tau$. Let now $A \subset X$ and $x_o \in X$ with $N_{x_o}^\tau(A) > \theta > 0$. Since τ^θ is completely regular, there exists a τ^θ-continuous function f from X to [0,1] such that $f(x_o) = 0, f(A^c) = 1$. For $x, y \in X$, define $F(x, y)$ on $\mathbf{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } \ t \leq 0 \\ 1-\theta & \text{if } \ |f(x) - f(y)| \geq t > 0 \\ 1 & \text{if } \ |f(x) - f(y)| < t \end{cases}$$

Clearly $F(x, y) = F(y, x) \in \mathbf{R}_\phi^+$ and $F(x, x) = \bar{0}$. We will prove that F satisfies the triangle inequality. So, assume that $F(x, y)(t_1) \wedge F(y, z)(t_2) > F(x, z)(t_1 + t_2)$. Then, $t_1, t_2 > 0, F(x, z)(t_1 + t_2) = 1 - \theta, F(x, y)(t_1) = F(y, z)(t_2) = 1$. Thus $t_1 > |f(x) - f(y)|, t_2 > |f(y) - f(z)|$ and hence $|f(x) - f(z)| < t_1 + t_2$, which implies that $F(x, z)(t_1 + t_2) = 1$, a contradiction. So F is a probabilistic pseudometric on X. Next we show that F is $\tau \times \tau$ continuous, or equivalently that $\tau_F \leq \tau$. So assume that $N_x^{\tau_F}(B) > r > 0$. Let $\theta_1 > r$ be such that $N_x^{\tau_F}(B) > \theta_1$. Choose $t > 0$ such that $1 - \sup_{y \notin B} F(x, y)(t) > \theta_1$ and so $F(x, y)(t) = 1 - \theta$ and $|f(x) - f(y)| \geq t$ if $y \notin B$. Thus $\{y : |f(x) - f(y)| < t\} \subset B$. This shows that B is a τ^θ-neighborhood of x. As $r < \theta$, B is a τ^r-neighborhood of x, i.e. $N_x^\tau(B) > r$ and so $\tau_F \leq \tau$. Finally if $y \notin A$, then $|f(y) - f(x_o)| = 1$ and so $F(y, x_o)(1/2) = 1 - \theta$, which implies that

$$N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^{\tau_F}(A) \geq 1 - \sup_{y \notin A} F(y, x_o)(1/2) \geq \theta.$$

This shows that $N_{x_o}^{\tau_{\mathcal{F}}} \geq N_{x_o}^\tau$ and so $\tau \leq \tau_{\mathcal{F}}$, which completes the proof of the implication (1) $\Rightarrow$ (2).
(3) $\Rightarrow$ (1) Assume that $\tau = \tau_{\mathcal{F}}$ for some family $\mathcal{F}$ of probabilistic pseudometrics on X. For each $F \in \mathcal{F}, \tau_F$ is completely regular and so $\tau_{\mathcal{F}}$ is completely regular since $\tau_{\mathcal{F}}^d = \bigvee_{F \in \mathcal{F}} \tau_F^d$ for each $0 \leq d < 1$. Hence the result follows.

We will denote by $[0,1]_\phi$ the subspace of $\mathbf{R}_\phi^+$ consisting of all $u \in \mathbf{R}_\phi^+$ with $u(t) = 1$ if $t > 1$.

Theorem 4.8 *A fuzzifying topological space (X, τ) is completely regular iff the following condition is satisfied: If $N_{x_o}(A) > \theta > 0$, then there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$.*

Proof: Assume that (X, τ) is completely regular and let $N_{x_o}(A) > \theta > 0$. Since τ^θ is completely regular, there exists $h : (X, \tau^\theta) \to [0,1]$ continuous, $h(x_o) = 0, h(y) = 1$ if $y \notin A$. For x, y in X, define $F(x,y)$ on $\mathbf{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1-\theta & \text{if } |h(x) - h(x_o)| \geq t > 0 \\ 1 & \text{if } |h(x) - h(x_o)| < t \end{cases}$$

Clearly $F(x,y) \in [0,1]_\phi$. Also $F(x,z) \preceq F(x,y) \oplus F(y,z)$. In fact, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > r > F(x,z)(t_1 + t_2)$. Then $t_1, t_2 > 0, F(x,y)(t_1) = F(y,z)(t_2) = 1$. Now $|h(x) - h(y)| < t_1, |h(y) - h(z)| < t_2$ and so $|h(x) - h(z)| < t_1 + t_2$ which implies that $F(x,z)(t_1 + t_2) = 1$, a contradiction. So F is a probabilistic pseudometric. Moreover F is $\tau \times \tau$ continuous, or equivalently $\tau_F \leq \tau$. In fact, let $N_x^{\tau_F}(B) > r > 0$. There exists a $t > 0$ such that $1 - \sup_{z \notin B} F(z,x)(t) > r$. If $z \notin B$, then $F(z,x)(t) < 1 - r < 1$ and so $F(z,x)(t) = 1 - \theta < 1 - r$, i.e. $r < \theta$, and $|h(z) - h(x)| \geq t$. Hence

$$M = \{z : |h(z) - h(x)| < t\} \subset B.$$

The set M is a τ^θ-neighborhood of x and hence a τ^r-neighborhood, i.e. $N_x^\tau(B) > r$. Thus $\tau \geq \tau_F$. Finally, define $f : X \to [0,1]_\phi, f(y) = F(y, x_o)$. Then f is τ-continuous, $f(x_o) = \bar{0}$. For $y \notin A$ and $0 < t < 1$, we have that $f(y)(t) = F(y, x_o)(t) = 1 - \theta$ (since $|h(x) - h(x_o| = 1 \geq t)$. Conversely, assume that the condition is satisfied and let $\mathcal{F}$ be the family of all $\tau \times \tau$ continuous pseudometrics on X. Then $\tau_{\mathcal{F}} \leq \tau$. Let $N_{x_o}^\tau(A) > \theta$. There exists a $\theta_1 > \theta$ such that $N_{x_o}^\tau(A) > \theta_1$. By our hypothesis, there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta_1$ if $y \notin A$ and $0 < t < 1$. Define $F(x,y) = D(f(x), f(y))$. Then F is $\tau \times \tau$ continuous and

$$\begin{aligned} N_{x_o}^{\tau_{\mathcal{F}}}(A) &\geq N_{x_o}^{\tau_F}(A) \geq 1 - \sup\nolimits_{y \notin A} F(x_o, y)(1) \\ &= 1 - \sup\nolimits_{y \notin A} D(\bar{0}, f(y))(1) \\ &= 1 - \sup\nolimits_{y \notin A} f(y)(1) \geq \theta_1 > \theta. \end{aligned}$$

Thus $N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^\tau(A)$, for every subset A of X and so $\tau \leq \tau_{\mathcal{F}}$. Therefore, $\tau = \tau_{\mathcal{F}}$ and so τ is completely regular.

For a fuzzifying topological space X, we will denote by $C(X, [0,1]_\phi)$ the family of all continuous functions from X to $[0,1]_\phi$.

Theorem 4.9 *A fuzzifying topological space (X, τ) is completely regular iff τ coincides with the weakest of all fuzzifying topologies τ_1 on X for which each $f \in C(X, [0,1]_\phi)$ is continuous.*

Proof: Assume that (X, τ) is completely regular and let τ_1 be the weakest of all fuzzifying topologies on X for which each $f \in C(X, [0,1]_\phi)$ is continuous. Clearly $\tau_1 \leq \tau$. On the other hand, let τ_2 be a fuzzifying topology on X for which each $f \in C(X, [0,1]_\phi)$ is continuous. Let $N_x^\tau(A) > \theta > 0$. In view of the preceding Theorem, there exists an $f \in C(X, [0,1]_\phi)$ such that $f(x) = \bar{0}, f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$. Let

$$G = \{u \in \mathbf{R}_\phi^+ : D(f(x), u)(1/2) = u(1/2) > 1 - \theta\}.$$

Then

$$N_{\bar{0}}(G) \geq 1 - \sup_{u \notin G} D(f(x), u)(1/2) \geq \theta.$$

Since f is τ_2-continuous, we have that $N_x^{\tau_2}(f^{-1}(G)) \geq \theta$. But $f^{-1}(G) \subset A$ since, for $y \notin A$, we have that $f(y)(1/2) = 1 - \theta$. Thus $N_x^{\tau_2}(A) \geq \theta$. This proves that $N_x^{\tau_2}(A) \geq N_x^{\tau}(A)$, for every subset A of X and so $\tau_2 \geq \tau$. This clearly proves that $\tau_1 = \tau$. Conversely, assume that $\tau_1 = \tau$. If σ is the usual fuzzifying topology of $\mathbf{R}_\phi^+$, then

$$\tau = \tau_1 = \bigvee_{f \in C(X,[0,1]_\phi)} f^{-1}(\sigma).$$

Since σ is completely regular, each $f^{-1}(\sigma)$ is completely regular and so τ is completely regular. This completes the proof.

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